Notes for the course REAL ANALYSIS

1 Topology

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neigbourhood* of x iff it contains an open set which in turn contains x.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U. U is called *dense* in S iff $\overline{U} = S$.

Exercise 1. Explain and justify the meaning of "smallest closed set" and "largest open set" in the above definition. Could one similarly talk about the "largest closed set" and the "smallest open set"?

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- *$\emptyset \in \mathcal{B}$.*
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exits a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6. Let S be a topological space and p a point in S. We call a family $\{U_\alpha\}_{\alpha\in A}$ of open neighbourhoods of p a *neighbourhood base* at p iff for any neighbourhood V of p there exists $\alpha \in A$ such that $U_\alpha \subseteq V$.

Definition 1.7 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous* iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open. We denote the space of continuous maps from S to T by $C(S,T)$.

Proposition 1.8. *Let S, T, U be topological spaces, $f \in C(S,T)$ and $g \in C(T,U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.9 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.10 (Product Topology)**.** Let S be the cartesian product $S = \prod_{\alpha\in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha\in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

Proposition 1.11. *Let S, T, X be topological spaces and $f \in C(S \times T, X)$. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x,y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neigbourhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x,y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. □

1.2 Special topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.12 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property*. We also say that S is a *Hausdorff space*.

Definition 1.13. Let S be a topologcial space. S is called *first-countable* iff there exists a countable neighbourhood base at each point of S. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.14 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subspace. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Definition 1.15. Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover. U is called *sequentially compact* iff every sequence in U contains a converging subsequence.

Proposition 1.16. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise**. □

Proposition 1.17. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise**. □

1.3 Sequences and convergence

Definition 1.18 (Convergence of sequences)**.** Let $x := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for evey neighbourhood U of x we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighbourhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n$: $x_k \in U$.

Proposition 1.19. *Let S be a topological space and $U \subseteq S$ a closed subspace. Let x be a sequence of points in U which has an accumulation point $p \in S$. Then, $p \in U$.*

Proof. Suppose $p \notin U$. Since U is closed $S \setminus U$ is an open neighbourhood of p. But $S \setminus U$ does not contain any point of x, so p cannot be accumulation point of x. This is a contradiction. □

Proposition 1.20. *Let S,T be topological spaces, $f \in C(S,T)$ and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in S converging to p. Then, the sequence $f\{(x_n)\}_{n \in \mathbb{N}}$ in T converges to $f(p)$.*

Proof. **Exercise**. □

Proposition 1.21. *Let S be Hausdorff space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in S which converges to a point $x \in S$. Then, $\{x_n\}_{n \in \mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise**. □

Definition 1.22 (Limit point compactness)**.** A topological space S is said to be *limit point compact* iff every sequence in S has an accumulation point (limit point).

Proposition 1.23. *A compact space is limit point compact.*

Proof. Consider a sequence x in a compact space S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighbourhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

Proposition 1.24. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable neighbourhood base $\{U_n\}_{n\in\mathbb{N}}$ at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighbourhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighbourhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighbourhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

2 Metric spaces

2.1 Basic Definitions

Definition 2.1. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x,y) = d(y,x) \quad \forall x,y \in S$. (symmetry)
- $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$. (triangle inequality)
- $d(x,y) = 0 \implies x = y \quad \forall x,y \in S$. (definiteness)

Then d is called a *metric* on S. S is also called a *metric space.*

Definition 2.2. If in the above definition the third condition is weakened to

- $d(x,x) = 0 \quad \forall x \in S$,

then d is called a *pseudometric* and S a *pseudometric space.*

Definition 2.3. Let S be a metric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x,y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x,y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 2.4. *Let S be a metric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is Hausdorff and first-countable.*

Proof. **Exercise**. □

Definition 2.5. A topological space is called *metrizable* iff there exists a metric such that the open balls given by the metric are a basis of its topology.

2.2 Completeness and Completion

Proposition 2.6. *Let S be a metric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 : d(x_n, p) < \epsilon$.*

Proof. Immediate. □

Definition 2.7. Let S be a metric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0 : d(x_n, x_m) < \epsilon$.

Proposition 2.8. *Any converging sequence in a metric space is a Cauchy sequence.*

Proof. **Exercise**. □

Proposition 2.9. *Suppose x is a Cauchy sequence in a metric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise**. □

Definition 2.10. Let S be a metric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U then U is called *complete*.

Proposition 2.11. *A complete subset of a metric space is closed. A closed subset of a complete metric space is complete.*

Proof. **Exercise**. □

Definition 2.12 (Totally boundedness)**.** Let S be a metric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 2.13. *A subset of a metric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.23) and hence (Proposition 2.9) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha \in A}$ of U that does not admit a finite subcovering. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k \in \mathbb{N}}$. Since U is complete the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open there exists $r > 0$ such that $B(p, r) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 2.14. *The notions of compactness and limit point compactness are equivalent in a metric space.*

Proof. **Exercise.** □

Proposition 2.15. *A totally bounded metric space is second-countable.*

Proof. **Exercise.** □

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

Exercise 2. Let S be a metric space.

- Let $x := \{x_n\}_{n \in \mathbb{N}}$ and $y := \{y_n\}_{n \in \mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n \to \infty} d(x_n, y_n)$ exists.
- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}^+_0$ by $\tilde{d}(x, y) := \lim_{n \to \infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.
- Show that $a \sim b \iff \tilde{d}(a, b) = 0$ defines an equivalence relation on T.

- Show that $\overline{S} := T/\sim$ is naturally a metric space.

- Show that $\overline{S}$ is complete. [Hint: First show that given a Cauchy sequence x in S and a subsequence x' of x we have $\tilde{d}(x, x') = 0$. That is, $x \sim y$ in T. Use this to show that for any Cauchy sequence x in S an equivalent Cauchy Sequence x' can be constructed which has a specific asymptotic behaviour. For example, x' can be made to satisfy $d(x'_n, x'_m) < \frac{1}{\min(m,n)}$. Now a Cauchy sequence $\hat{x} = \{\hat{x}^n\}_{n\in\mathbb{N}}$ in $\overline{S}$ consists of equivalence classes $\hat{x}^n$ of Cauchy sequences in S. Given some representative x^n of $\hat{x}^n$ show that there is another representative x'^n whith specific asymptotic behaviour. Using such representatives x'^n for all $n \in \mathbb{N}$ show that the equivalence class in $\overline{S}$ of the diagonal sequence $y := \{x'^n_n\}_{n\in\mathbb{N}}$ is a limit of $\hat{x}$.]

- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 2.16. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 2.17 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric embedding. Then, there is a unique isometric embedding $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **Exercise.** □